Non-relativistic Maxwell-Chern-Simons Vortices

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Abstract. *The non-relativistic Maxwell-Chern-Simons model recently introduced by Manton is shown to admit self-dual vortex solutions with non-zero electric field. The interrelated "geometric" and "hidden" symmetries are explained. The theory is also extended to (non-relativistic) spinors. A relativistic, self-dual model, whose non-relativistic limit is the Manton model is also presented. The relation to previous work is discussed.*

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1. Introduction

In a recent paper [1], Manton proposed a modified version of the Landau-Ginzburg model for describing Type II superconductivity. His Lagrange density is a subtle mixture blended from the standard Landau-Ginzburg expression, augmented with the Chern-Simons term:

$$\begin{aligned}\mathcal{L} = &-\frac{1}{2}B^2 + \gamma\frac{i}{2}\big(\phi^* D_t\phi - \phi(D_t\phi)^*\big) - \frac{1}{2}\big|\vec{D}\phi\big|^2 - \frac{\lambda}{8}\big(1-|\phi|^2\big)^2 \\ &+ \mu\big(Ba_t + E_2a_1 - E_1a_2\big) - \gamma a_t - \vec{a}\cdot\vec{J}^T,\end{aligned} \tag{1.1}$$

where μ, $\gamma > 0$, $\lambda > 0$ are constants, $D_t\phi = \partial_t\phi - ia_t\phi$, $D_i\phi = \partial_i\phi - ia_i\phi$, $B = \partial_1 a_2 - \partial_2 a_1$ is the magnetic field and $\vec{E} = \vec{\nabla}a_t - \partial_t\vec{a}$ is the electric field. This Lagrangian differs from the standard expression in that (i) it is linear in $D_t\phi$; (ii) the electric term $\vec{E}^2$ is missing; (iii) it includes the terms $-\gamma a_t$ and $-\vec{a}\cdot\vec{J}^T$, where $\vec{J}^T$ is the (constant) transport current. The properties (i) and (ii) come from the requirement of Galilean rather than Lorentz invariance [2]. The term $-\gamma a_t$ results in modifying the Gauss law (eqn. (1.4) below); the term $-\vec{a}\cdot\vec{J}^T$ is then needed in order to restore the Galilean invariance. To be so, the transport current has to transform as $\vec{J}^T \to \vec{J}^T + \gamma\vec{v}$ under a Galilei boost [1].

The field equations derived from (1.1) are

$$i\gamma D_t\phi = -\frac{1}{2}\vec{D}^2\phi - \frac{\lambda}{4}\big(1-|\phi|^2\big)\phi, \tag{1.2}$$

$$\epsilon_{ij}\partial_j B = J_i - J_i^T + 2\mu\,\epsilon_{ij}\,E_j, \tag{1.3}$$

$$2\mu B = \gamma\big(1-|\phi|^2\big), \tag{1.4}$$

where the (super)current is $\vec{J} = (1/2i)\big(\phi^*\vec{D}\phi - \phi(\vec{D}\phi)^*\big)$. The matter field satisfies hence a gauged, planar non-linear Schrödinger equation. The second equation is Ampère's law without the displacement current, as usual in the "magnetic-type" Galilean electricity [2]. The last equation called the Gauss law is the (modified) "Field-Current Identity" of Jackiw and Pi [3].

Conventional Landau-Ginzburg theory admits finite-energy, static, purely magnetic vortex solutions [4]. For a specific value of the coupling constant, one can find solutions by solving instead the first-order "Bogomolny" equations [5],

$$\begin{aligned}&(D_1 + iD_2)\phi = 0, \\ &2B = 1 - |\phi|^2.\end{aligned} \tag{1.5}$$

Now, as observed by Manton, these same solutions yield magnetic vortices with $a_t = 0$, also in his model, when $\vec{J}^T = 0$, $\lambda = 1$ and $\mu = \gamma$. Manton also conjectures the existence of further solutions with a non-vanishing electric field.

In this Paper, we show that this is indeed the case : a slight generalization of the self-duality equations (1.5) does indeed provide self-dual vortices with nonzero electric field. Being self-dual, these solutions are stable.

Next, using the equivalence of the model with one in constant external electric and magnetic fields, we discuss the subtle symmetries. In Section 4, we construct self-dual non-relativistic spinorial vortices along the same lines. In Section 5 we present a self-dual, relativistic model of the same type, whose non-relativistic limit is the Manton model. Finally, we compare our results to those obtained by other people.

2. Self-dual vortices

In the frame where $\vec{J}^T = 0$ (which can always be achieved by a Galilei boost), the static Manton equations (1.2-4) read

$$\begin{aligned} &\gamma a_t \phi = -\frac{1}{2}\vec{D}^2\phi - \frac{\lambda}{4}(1-|\phi|^2)\phi,\\ &\vec{\nabla}\times B = \vec{J} + 2\mu\vec{\nabla}\times a_t,\\ &2\mu B = \gamma\big(1-|\phi|^2\big). \end{aligned} \tag{2.1}$$

Let us try to solve these by the first-order Ansatz

$$\begin{aligned} &(D_1 \pm iD_2)\phi = 0,\\ &2\mu B = \gamma\big(1-|\phi|^2\big). \end{aligned} \tag{2.2}$$

From the first of these relations we infer that $\vec{D}^2 = \mp i\big[D_1, D_2\big] = \mp B$ and $\vec{J} = \mp\frac{1}{2}\vec{\nabla}\times\varrho$, where $\varrho = |\phi|^2$. Inserting into the non-linear Schrödinger equation we find that it is identically satisfied when $a_t = (\pm 1/4\mu - \lambda/4\gamma)(1-\varrho)$. Then from Ampère's law we get that λ has to be

$$\lambda = \pm 2\frac{\gamma}{\mu} - \frac{\gamma^2}{\mu^2}. \tag{2.3}$$

The scalar potential is thus

$$a_t = \tfrac{1}{4\mu}\big(\mp 1 + \tfrac{\gamma}{\mu}\big)\big(1-\varrho\big). \tag{2.4}$$

Then the vector potential is expressed using the "self-dual" (SD) Ansatz (2.2) as

$$\vec{a} = \pm\tfrac{1}{2}\vec{\nabla}\times\log\varrho + \vec{\nabla}\omega, \tag{2.5}$$

where ω is an arbitrary real function chosen so that $\vec{a}$ is regular [3]. Inserting this into the Gauss law, we end up with the "Liouville-type" equation

$$\triangle\log\varrho = \pm\alpha\big(\rho - 1\big), \qquad \alpha = \frac{\gamma}{\mu}.$$

Now, if we want a "confining" (stable) and lower-bounded scalar potential, λ has to be positive. Then we see from eq. (2.3) that for the upper sign this means $0 < \alpha < 2$, whereas for the lower sign $-2 < \alpha < 0$. In any of the two cases (α positive or negative), the coefficient of $(\rho - 1)$ in the r. h. s. is always positive: in the upper sign, it is α with $\alpha > 0$, in the lower sign, it is $-\alpha$ with $\alpha < 0$. We consider henceforth the equation

$$\triangle \log \varrho = |\alpha|\big(\rho - 1\big), \tag{2.6}$$

Note that the electric field, $\vec{E} = \vec{\nabla} a_t$, only vanishes for $\mu = \pm\gamma$, i.e., when $\lambda = 1$, which is Manton's case.

Before analyzing the solutions of Eq. (2.6), let us discuss the finite-energy conditions. As it will be derived in the next Section, in the frame where $\vec{J}^T = 0$, the energy associated to the Lagrangian (1.1) is

$$H = \int \Big\{ \tfrac{1}{2}\big|\vec{D}\phi\big|^2 + \tfrac{1}{2}B^2 + U(\phi)\Big\}\, d^2\vec{x}, \qquad U(\phi) = \frac{\lambda}{8}\big(1 - |\phi|^2\big)^2. \tag{2.7}$$

Eliminating the magnetic term $B^2/2$ by using the Gauss law (1.4) results in shifting merely the coefficient of the non-linear term,

$$H = \int \Big\{ \tfrac{1}{2}\big|\vec{D}\phi\big|^2 + \frac{\Lambda}{8}\big(1 - |\phi|^2\big)^2\Big\}\, d^2\vec{x}, \qquad \Lambda = \lambda + \frac{\gamma^2}{\mu^2}. \tag{2.8}$$

Finite energy requires, just like in the Landau-Ginzburg case, $\vec{D}\phi \to 0$ and $|\phi|^2 \to 1$ so that our objects represent *topological vortices*: The first of these equations implies that the angular component of vector potential behaves asymptotically as n/r. The integer n here is also the winding number of the mapping defined by the asymptotic values of ϕ into the unit circle,

$$n = \frac{1}{2\pi}\oint_S \vec{a}\cdot d\vec{\ell} = \frac{1}{2\pi}\int B\, d^2\vec{x}, \tag{2.9}$$

so that the magnetic flux is necessarily quantized, and is related to the particle number

$$N \equiv \int \big(1 - |\phi|^2\big)\, d^2\vec{x} = \frac{2\mu}{\gamma}\int B\, d^2\vec{x} = 4\pi\big(\frac{\mu}{\gamma}\big)n. \tag{2.10}$$

N is conserved since the supercurrent satisfies the continuity equation $\partial_t\varrho + \vec{\nabla}\cdot\vec{J} = 0$.

Not surprisingly, our self-duality equations (2.2) can also be obtained by studying the energy, (2.7). Using the identity $\big|\vec{D}\phi\big|^2 = \big|(D_1 \pm iD_2)\phi\big|^2 \pm B|\phi|^2 \pm \vec{\nabla}\times\vec{J}$ and assuming that the fields vanish at infinity, the integral of the current-term can be dropped, so that H becomes

$$\int \Big\{ \frac{1}{2}\Big|(D_1 \pm iD_2)\phi\Big|^2 + \Big[\big(\mp\frac{\gamma}{4\mu} + \frac{\Lambda}{8}\big)(1 - |\phi|^2)^2\Big]\Big\} d^2\vec{x} \pm \underbrace{\tfrac{1}{2}\int B\, d^2\vec{x}}_{\pi n}, \tag{2.11}$$

which shows that the energy is positive definite when the square bracket vanishes, i.e., for the chosen potential with λ as in Eq. (2.3). In this case, the energy admits a lower "Bogomolny" bound, $H \geq \pi|n|$, with the equality only attained when the SD equations hold.

Eqn. (2.6) is similar to that of Bogomolny in the Landau-Ginzburg theory [5] to which it reduces when $|\alpha| = 1$. The proofs of Weinberg, and of Taubes [6], carry over literally to show, for each n, the existence of a $2n$-parameter family of solutions. Radial solutions were studied numerically by Barashenkov and Harin [7]. The solutions behave as in the Bogomolny case. Write $\phi = f(r)e^{in\theta}$. Linearizing the SD Eqns. (2.2), for $\varphi = 1 - f$ we get

$$\varphi'' + \frac{1}{r}\varphi' - |\alpha|\varphi = 0, \tag{2.12}$$

which is Bessel's equation of order zero. The solution and its asymptotic behaviour are therefore

$$\varphi(r) \quad \sim \quad K_0(mr) \quad \sim \quad \frac{C}{\sqrt{r}}e^{-mr}, \qquad\qquad m = \sqrt{|\alpha|}. \tag{2.13}$$

The magnetic and electric fields behave in turn as

$$B = \frac{\alpha}{2}(1-f^2) \sim \alpha\frac{D}{\sqrt{r}}e^{-mr}, \qquad \vec{E} = -\tfrac{1}{4\mu}\big(\mp 1 + \alpha\big)\,\vec{\nabla}f^2 \sim \frac{G}{\sqrt{r}}e^{-mr}. \tag{2.14}$$

3. Symmetries and conserved quantities

Let us remember the definition: an infinitesimal transformation, represented by a vector field X^μ on space-time, is a symmetry when it changes the Lagrangian by a surface term,

$$\mathcal{L} \to \mathcal{L} + \partial_\alpha K^\alpha \tag{3.1}$$

for some function K. To each such transformation, Nœther's theorem associates a conserved quantity, namely

$$C = \int \left(\frac{\delta\mathcal{L}}{\delta(\partial_t\chi)}\delta\chi - K^t\right) d^2\vec{x}, \tag{3.2}$$

where χ denotes collectively all fields in the theory [8].

Our clue for understanding the symmetry properties of the Manton system is to observe that setting

$$B^{ext} \equiv \frac{\gamma}{2\mu}, \qquad E_k^{ext} = -\frac{\epsilon_{kl}{J^T}_l}{2\mu}, \tag{3.3}$$

the equations of motion (1.2-4) become

$$
\begin{aligned}
& i\gamma D_t\phi = -\frac{1}{2}\vec{D}^2\phi - \frac{\lambda}{4}\big(1-|\phi|^2\big)\phi, \\
& \epsilon_{ij}\partial_j\widetilde{B} = J_i + 2\mu\,\epsilon_{ij}\,\widetilde{E}_j, \\
& 2\mu\widetilde{B} = -\gamma|\phi|^2,
\end{aligned}
\tag{3.4}
$$

where $\widetilde{B} = B - B^{ext}$, $\widetilde{E}_i = E_i - E_i^{ext}$, and $D_\alpha = \partial_\alpha - ia_\alpha$, where $a_\alpha = \tilde{A}_\alpha + A_\alpha^{ext}$, so that $\widetilde{F}_{\alpha\beta} = \partial_\alpha\widetilde{A}_\beta - \partial_\beta\widetilde{A}_\alpha$. These equations describe a non-relativistic scalar field with Maxwell-Chern-Simons dynamics and an external, constant electromagnetic field [9], [10].

The equivalence of the two models can also be checked on the respective Lagrangians. That of the external-field problem is in fact

$$
\begin{aligned}
\mathcal{L}^{ext} = &-\frac{1}{2}B^2 + \gamma\frac{i}{2}\big(\phi^* D_t\phi - \phi(D_t\phi)^*\big) - \frac{1}{2}\big|\vec{D}\phi\big|^2 - U(|\phi|) \\
&+ \mu\left(\tilde{B}\tilde{A}_t + \tilde{E}_2\tilde{A}_1 - \tilde{E}_1\tilde{A}_2\right).
\end{aligned}
\tag{3.5}
$$

Inserting here $A_k^{ext} = -\gamma\epsilon_{kl}x^l/4\mu$ and $A_t^{ext} = 0$, one gets up to surface terms the Manton Lagrangian (1.1), without the transport current term. This latter is finally recovered when applying a galilean boost,

$$
\phi(\vec{x},t) \to e^{-i(\vec{J}^T\cdot\vec{x}+\frac{1}{2}|\vec{J}^T|^2 t/\gamma)}\,\phi\big(\vec{x}+\vec{J}^T t/\gamma, t\big).
\tag{3.6}
$$

The necessity of adding a transport current corresponds hence to the arisal of an electric field under a boost.

Before studying the symmetries of the Manton system, let us recall that Jackiw and Pi have shown in Ref. [3] that a pure Chern-Simons-matter system with the non - symmetry - breaking potential $U = -(g/2)|\phi|^4$ admits the Schrödinger group as symmetry. The "geometric" action on space-time of this latter is generated by the 8-parameter vectorfield

$$
\begin{pmatrix} X^t \\ \vec{X} \end{pmatrix} = \begin{pmatrix} -\chi t^2 - \delta\, t - \epsilon \\ \Omega(\vec{x}) - \big(\frac{1}{2}\delta + \chi t\big)\,\vec{x} + t\vec{\beta} + \vec{\gamma} \end{pmatrix},
\tag{3.7}
$$

where $\Omega \in \mathrm{so}(2)$, $\vec{\beta}, \vec{\gamma} \in \mathbf{R}^2$, $\epsilon, \chi, \delta \in \mathbf{R}$, interpreted as rotation, boost, space translation, time translation, expansion, dilatation.

When this system is put into an external field, only those symmetries remain which are symmetries for this latter in the sense of Ref. [8]:

$$
X^\alpha F_{\alpha\beta}^{ext} = \partial_\beta\Psi.
\tag{3.8}
$$

For a constant electric and magnetic field, it is readily seen that the Schrödinger symmetry, *acting as in (3.7) on spacetime*, is broken. Only the time and space translations survive in general: Eqn. (3.8) is satisfied with

$$
\begin{aligned}
&\Psi = \vec{x}\cdot\vec{E}^{ext}\epsilon && \text{for time translations,}\\
&\Psi = B^{ext}\vec{\gamma}\times\vec{x} + t\vec{\gamma}\cdot\vec{E}^{ext} && \text{for space translations.}
\end{aligned}
\tag{3.9}
$$

Exceptions may also occur, namely,

- When $B^{ext}=0$, we also have boosts;
- When $\vec{E}^{ext}=0$, we also have rotations.

In what follows, we only consider the case $B^{ext}\neq 0$.

Now the pure Chern-Simons system in a constant external electromagnetic field admits another, "hidden" symmetry [9], [10], [11], with generators

$$
\begin{aligned}
&\cos\omega t\begin{pmatrix}0\\ \cos\omega t\,\gamma_1-\sin\omega t\,\gamma_2\\ \sin\omega t\,\gamma_1+\cos\omega t\,\gamma_2\end{pmatrix} && \gamma_1,\ \gamma_2\in\mathbf{R} && \text{“translations”},\\
&-\frac{\sin\omega t}{\omega}\begin{pmatrix}0\\ \cos\omega t\,\beta_1-\sin\omega t\,\beta_2\\ \sin\omega t\,\beta_1+\cos\omega t\,\beta_2\end{pmatrix} && \beta_1,\ \beta_2\in\mathbf{R} && \text{“boosts”},\\
&\begin{pmatrix}0\\ -\Omega\big(x_2-(E_1^{ext}/B^{ext})t\big)\\ \Omega\big(x_1+(E_2^{ext}/B^{ext})t\big)\end{pmatrix} && \Omega\in\mathbf{R} && \text{“rotations”},
\end{aligned}
\tag{3.10}
$$

(where $\omega=\frac{1}{2}B^{ext}$), as well 3 more generators, we call "dilatations", "expansions" and "time translations". (Their rather complicated expressions are here omitted, since they are not needed for our purposes). Surprisingly, this algebra turns out to be abstractly isomorphic to the Schrödinger algebra, as anticipated by the terminology. The action of this "hidden" Schrödinger algebra on spacetime is different from the geometric one in (3.7), though. The "hidden rotations" look, e.g., rather as cycloidal motions; they reduce to the "ordinary" ("geometric") rotations only when $\vec{E}^{ext}=0$. Note also that the geometric translations in (3.7) are related to the "hidden" ones in (3.10) according to

$$
\begin{aligned}
&(\text{geometric translation})_i =\\
&(\text{“hidden translation”})_i+\omega\epsilon_{ij}\,(\text{“hidden boost”})_j.
\end{aligned}
\tag{3.11}
$$

Let us now return to the Manton system. The "non-relativistic Maxwell term" B^2 is Schrödinger invariant. The potential $(\lambda/4)\big(1-|\phi|^2\big)^2$ breaks, however, the "hidden" dilatations, expansions and even time translations (that's why we did not write them at all). We are hence left with a five-parameter subgroup of the "hidden" Galilei group made of ("hidden") "translations", "boosts" and "rotations", which acts as symmetry for the Manton system. (Note that the new, "geometric" time translations do *not* belong to this unbroken subgroup).

Having determined the symmetries of our problem, we now turn to the associated conserved quantities. The formula (3.2) can also be written in terms of the energy-momentum tensor $T_{\alpha\beta}$ as

$$C_X=\int T_{0\alpha}X^\alpha d^2\vec{x}. \tag{3.12}$$

The energy-momentum tensor $T_{\alpha\beta}$ is readily derived by modifying the expression found by Jackiw and Pi in the pure Chern-Simons case [3]. Returning to the original variables, we find:

(3.13)

$$\begin{aligned}
T_{00}&=\tfrac{1}{2}\big|\vec{D}\phi\big|^2+\frac{\Lambda}{8}\big(1-|\phi|^2\big)^2+\vec{J}^T\cdot\vec{a}-\tfrac{1}{2}|\vec{J}^T|^2|\phi|^2, \qquad \Lambda=\lambda+\big(\frac{\gamma}{\mu}\big)^2,\\
T_{k0}&=-\tfrac{1}{2}\big[(D_t\phi)^*D_k\phi+(D_t\phi)(D_k\phi)^*\big]+\epsilon_{kj}E_jB-a_0J^T_k-\tfrac{1}{2\gamma}|\vec{J}^T|^2J_k,\\
T_{0k}&=-\gamma\tfrac{i}{2}\big(\phi^*D_k\phi-\phi(D_k\phi)^*\big)+\gamma a_k-\gamma J^T_k|\phi|^2,\\
T_{ij}&=\tfrac{1}{2}\Big((D_i\phi)^*D_j\phi+(D_j\phi)^*D_i\phi-\delta_{ij}\big|\vec{D}\phi\big|^2\Big)+J^T_i a_j-J_iJ^T_j,\\
&\quad+\tfrac{1}{4}\big(\delta_{ij}\bigtriangleup-2\partial_i\partial_j\big)\varrho+\delta_{ij}\left[T_{00}-\gamma a_0-2\vec{a}\cdot\vec{J}^T+\frac{\lambda}{4}\big(|\phi|^2-1\big)+\tfrac{1}{2}|\vec{J}^T|^2|\phi|^2\right].
\end{aligned}$$

(The energy-momentum tensor has been improved so that the integrals below converge). It satisfies the continuity equation $\partial_tT_{0\mu}+\partial_kT_{k\mu}=0$. Note that $T_{0j}\neq T_{j0}$ and that T_{ij} is only symmetric in the frame where $\vec{J}^T=0$; this is obviously related to the breaking of "ordinary" rotational symmetry.

Note that, unlike in a relativistic theory, the energy-momentum tensor is not traceless but satisfies instead

$$\sum_i T_{ii}=2T_{00}+\frac{\lambda}{2}\big(|\phi|^2-1\big)-2\gamma a_0-3\vec{J}^T\cdot\vec{a}-\vec{J}\cdot\vec{J}^T+|\vec{J}^T|^2|\phi|^2. \tag{3.14}$$

(This is consistent with the breaking of the Schrödinger symmetry).

For the surviving geometric symmetries we find the conserved energy and momentum,

$$\begin{aligned} H &= \int \Big\{ \tfrac{1}{2}\big|\vec{D}\phi\big|^2 - \tfrac{1}{2}\big|\vec{J}^T\big|^2|\phi|^2 + \frac{\Lambda}{8}\big(1-|\phi|^2\big)^2 - \big(\vec{x}\times\vec{J}^T\big)B \Big\}\, d^2x, \\ \Lambda &= \lambda + \big(\tfrac{\gamma}{\mu}\big)^2, \qquad \text{energy} \\ \mathcal{P}_i &= \gamma \int \Big\{ J_i - J_i^T|\phi|^2 + \epsilon_{ij}\big(x^j - t\frac{J_j^T}{\gamma}\big)B \Big\}\, d^2\vec{x} \qquad \text{momentum} \end{aligned} \tag{3.15}$$

The energy integral converges, since $|\vec{D}\phi| \to |\vec{J}^T|$ and $|\phi|^2 \to 1$ when $|\vec{x}| \to \infty$. Note also the extra piece proportional to the magnetic field B in the momentum. Because of this piece, the Dirac bracket of the momenta satisfies

$$\big\{\mathcal{P}_1, \mathcal{P}_2\big\} = \gamma \int B d^2x = \gamma\, 2\pi\, n, \tag{3.16}$$

rather then vanishes (see the Appendix). (This has been found also by Barashenkov and Harin [7] in their model).

The conserved quantities associated to the the unbroken part of the "hidden" symmetry (3.10) can also be readily calculated using (3.12). The explicit expressions are not illuminating and therefore omitted. It is, however, interesting to point out the relation between the "geometric" momentum, $\vec{\mathcal{P}}$, the "hidden momentum" $\vec{p}$, and "hidden boost" $\vec{g}$,

$$\mathcal{P}_i = p_i + \omega\epsilon_{ij}g_j, \qquad \omega \equiv \tfrac{1}{2}B^{ext}, \tag{3.17}$$

which is plainly the analog of the relation (3.11) between the generating vectorfields. This explains the unusual commutation relations (3.16). Our hidden "translations" and "boosts" satisfy in fact

$$\big\{p_i, p_j\big\} = 0 = \big\{g_i, g_j\big\}, \qquad \big\{g_j, p_i\big\} = \gamma\, N\delta_{ij}, \tag{3.18}$$

where N is the particle number (2.10).

For the "hidden angular momentum", we get in turn

$$\begin{aligned} M = \gamma \int \Big\{ & \vec{x}\times\big(\vec{J} - \vec{J}^T|\phi|^2\big) - \tfrac{1}{2}r^2 B \\ & -\tfrac{t}{\gamma}\Big[\vec{J}^T\times\vec{J} - (\vec{x}\cdot\vec{J}^T)B\Big] - \tfrac{1}{2}\big(\tfrac{t}{\gamma}\big)^2|\vec{J}^T|^2 B \Big\}\, d^2\vec{x} \quad \text{“angular momentum”.} \end{aligned} \tag{3.19}$$

Note here the extra piece proportional to the total magnetic field B. Note also that the integrals converge since $\vec{J} \to \vec{J}^T$, and $|\phi| \to 1$ at spatial infinity. For $\vec{J}^T = 0$, (3.19) reduces to the well-known formulæ in a magnetic field.

4. Spinor vortices

In Ref. [12], we found non-relativistic, spinor vortices in pure Chern-Simons theory. Below we generalize our construction to the magnetic type non-relativistic Maxwell-Chern-Simons theory of Manton's type. Let Φ denote a 2-component Pauli spinor. We posit the following equations of motion.

$$\begin{cases} i\gamma D_t\Phi = -\frac{1}{2}\big[\vec{D}^2 + B\sigma_3\big]\Phi & \text{Pauli eqn.} \\ \epsilon_{ij}\partial_j B = J_i - J_i^T + 2\mu\,\epsilon_{ij}\,E_j & \text{Ampère's eqn.} \\ 2\mu B = \gamma\big(1 - |\Phi|^2\big) & \text{Gauss' law} \end{cases} \tag{4.1}$$

where the current is now

$$\vec{J} = \frac{1}{2i}\Big(\Phi^\dagger\vec{D}\Phi - (\vec{D}\Phi)^\dagger\Phi\Big) + \vec{\nabla}\times\Big(\frac{1}{2}\,\Phi^\dagger\sigma_3\Phi\Big). \tag{4.2}$$

The system is plainly non-relativistic, and it admits self-dual vortex solutions, as we show now. The transport current can again be eliminated by a galilean boost. For fields which are static in the frame where $\vec{J}^T = 0$, the equations of motion become

$$\begin{cases} \big[\frac{1}{2}(\vec{D}^2 + B\sigma_3) + \gamma a_t\big]\Phi = 0, \\ \vec{\nabla}\times B = \vec{J} + 2\mu\,\vec{\nabla}\times a_t, \\ 2\,\dfrac{\mu}{\gamma}B = 1 - \Phi^\dagger\Phi. \end{cases} \tag{4.3}$$

Let us now attempt to solve the static equations (4.3) by the first-order Ansatz

$$\big(D_1 \pm iD_2\big)\Phi = 0. \tag{4.4}$$

Then

$$\vec{D}^2 = \mp B \qquad \text{and} \qquad \vec{J} = \tfrac{1}{2}\vec{\nabla}\times\Big[\Phi^\dagger(\mp 1 + \sigma_3)\Phi\Big], \tag{4.5}$$

so that the static Pauli equation requires

$$\Big[(\mp 1 + \sigma_3)B + 2\gamma a_t\Big]\Phi = 0. \tag{4.6}$$

Let us decompose Φ into chiral components,

$$\Phi = \Phi_+ + \Phi_-, \qquad \text{where} \qquad \Phi_+ = \begin{pmatrix} 0 \\ \chi \end{pmatrix} \qquad \text{and} \qquad \Phi_- = \begin{pmatrix} \varphi \\ 0 \end{pmatrix}. \tag{4.7}$$

Eqn. (4.6) requires that Φ has a definite chirality. One possibility would be $\Phi_+ = 0$ for the upper sign and $\Phi_- = 0$ for the lower sign, as in Ref. [12]. In both cases, a_t would have to vanish. It is, however, easely seen to be inconsistent with Ampère's law.

Curiously, there is another possibility: one can have

$$a_t = \pm\tfrac{1}{\gamma}\,B \qquad \text{and} \qquad \begin{array}{lll} \Phi_- = 0 & \text{i.e. } \Phi \equiv \Phi_+ & \text{for the upper sign} \\ \Phi_+ = 0 & \text{i.e. } \Phi \equiv \Phi_- & \text{for the lower sign} \end{array}. \tag{4.8}$$

Then $\vec{J} = \mp\vec{\nabla}\times\big|\Phi_\pm\big|^2$, so that Ampère's law requires

$$\vec{\nabla}\times\Big(\big[1\mp\tfrac{2\mu}{\gamma}\big]B \pm \big|\Phi_\pm\big|^2\Big) = 0. \tag{4.9}$$

But now $\big|\Phi_\pm\big|^2 = \big|\Phi\big|^2$, which is $1-(2\mu/\gamma)B$ by the Gauss law, so that (4.9) holds when

$$\alpha \equiv \pm\frac{\gamma}{\mu} = 4. \tag{4.10}$$

In conclusion, for the particular value (4.10), the second-order field equations can be solved by solving one or the other of the first-order equations in (4.4). Now these latter conditions fix the gauge potential as

$$\vec{a} = \pm\tfrac{1}{2}\vec{\nabla}\times\log\varrho, \qquad \varrho \equiv \big|\Phi\big|^2 = \big|\Phi_\pm\big|^2. \tag{4.11}$$

Then the Gauss law yields

$$\triangle\log\varrho = 4(\varrho-1), \tag{4.12}$$

which is again the "Liouville-type" equation (2.6) we studied before. Note that the sign — the same for both choices — is automatically positive and equal to 4.

The equations of motion (4.1) can be derived from the Lagrangian

$$\begin{aligned} \mathcal{L} = &-\frac{1}{2}B^2 + \frac{i\gamma}{2}\big[\Phi^\dagger(D_t\Phi) - (D_t\Phi)^\dagger\Phi\big] - \frac{1}{2}(\vec{D}\Phi)^\dagger(\vec{D}\Phi) \\ &+ \frac{B}{2}\Phi^\dagger\sigma_3\Phi + \mu\big(Ba_t + E_2a_1 - E_1a_2\big) - \gamma a_t - \vec{a}\cdot\vec{J}^T. \end{aligned} \tag{4.13}$$

Then, in the frame where $\vec{J}^T = 0$, the associated energy is

$$H = \frac{1}{2}\int\Big\{B^2 + \big|\vec{D}\Phi\big|^2 - B\,\Phi^\dagger\sigma_3\Phi\Big\}\,d^2\vec{x}. \tag{4.14}$$

Using the identity

$$\big|\vec{D}\Phi\big|^2 = \big|(D_1 \pm iD_2)\Phi\big|^2 \pm B\,\Phi^\dagger\Phi \tag{4.15}$$

(up to surface terms), the energy is rewritten as

$$H = \tfrac{1}{2}\int\Big\{B^2 + \big|(D_1 \pm iD_2)\Phi\big|^2 - B\Big[\Phi^\dagger(\mp 1 + \sigma_3)\Phi\Big]\Big\}\, d^2\vec{x}.$$

Eliminating B using the Gauss law, we get finally, for purely chiral fields, $\Phi = \Phi_\pm$,

$$H = \tfrac{1}{2}\int\Big\{\big|(D_1 \pm iD_2)\Phi_\pm\big|^2 + \frac{\gamma}{4\mu}\big[\mp 4 + \frac{\gamma}{\mu}\big]\big(1 - |\Phi_\pm|^2\big)^2\Big\}\, d^2\vec{x} \pm \int B\, d^2\vec{x}. \tag{4.16}$$

Here the last integral yields the topological charge $\pm 2\pi n$. The integral is positive definite when $\pm\gamma/\mu \geq 4$ depending on the chosen sign, yielding the Bogomolny bound $H \geq 2\pi|n|$. The Pauli term results hence in *doubling* the Bogomolny bound with respect to the scalar case. The bound can be saturated when $\pm\gamma/\mu = 4$ and the self-dual equations (4.4) hold.

5. Relativistic models and their non-relativistic limit

In relativistic Maxwell-Chern-Simons theory self-dual solutions only arise when an auxiliary neutral field N is added [13]. Here we present a model of the type considered by Lee, Lee and Min, which (i) is relativistic; (ii) can be made self-dual; (iii) its non-relativistic limit is the Manton model presented in this paper. Let us consider in fact $(1+2)$-dimensional Minkowski space with the metric $(c^2/\gamma, -1, -1)$ where $\gamma > 0$ is a constant. Let us chose the Lagrangian

$$\mathcal{L}_R = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{\mu}{2}\epsilon^{\mu\nu\rho}F_{\mu\nu}a_\rho + \big(D_\mu\psi\big)\big(D^\mu\psi\big)^* + a^\mu {J^T}_\mu + \tfrac{\gamma}{2c^2}\partial_\mu N\partial^\mu N - V. \tag{5.1}$$

Here N is an auxiliary neutral field, which we chose real. We have also included the term $a^\mu {J^T}_\mu$ where the Lorentz vector ${J^T}_\mu$ represents the relativistic generalization of Manton's transport current. We chose ${J^T}_\mu$ to be time-like, $\mathrm{I}^2 \equiv \frac{\gamma}{c^2}{J^T}_\mu {J^T}^\mu > 0$. Our choice for the potential is

$$V = \frac{\beta}{2}\big(|\psi|^2 - 2|\mu|N - \frac{\mathrm{I}}{2m\gamma}\big)^2 + \frac{\gamma}{c^2}\big(N + mc^2\big)^2|\psi|^2 - (N + mc^2)\mathrm{I}, \tag{5.2}$$

where $\beta > 0$. Although the potential is *not* positive definite, this will cause no problem when the Gauss law is taken into account, as it will be explained later. Note that a similar behaviour has already been encountered before [14]. This Lagrangian is clearly Lorentz-invariant so that the model is indeed relativistic.

The associated equations of motion read

$$\begin{aligned}
&D_\mu D^\mu\psi + \frac{\partial V}{\partial\psi^*} = 0, && \text{Non-linear Klein-Gordon eqn.}\\
&\tfrac{\gamma}{c^2}\partial_0 F_{0i} + \epsilon_{ij}\partial_j F_{12} + 2\mu\epsilon_{ij}F_{0j} - J_i + {J^T}_i = 0, && \text{Ampère's law}\\
&\tfrac{\gamma}{c^2}\partial_i F_{0i} + 2\mu F_{12} = \tfrac{\gamma}{c^2}\big(J_0 - {J^T}_0\big), && \text{Gauss' law}\\
&\tfrac{\gamma}{2c^2}\partial_\mu\partial^\mu N + \frac{\partial V}{\partial N} = 0 && \text{auxiliary eqn. for N.}
\end{aligned} \tag{5.3}$$

One can always choose a Lorentz frame where the spatial components of the transport current vanishes, ${J^T}_\mu = (-\frac{c^2}{\gamma}\mathrm{I}, 0)$. In such a frame, using the Gauss law, for the energy we find

$$H_R = \int \Big\{ \tfrac{\gamma}{2c^2}\,\vec{E}^2 + \tfrac{1}{2}B^2 + \tfrac{\gamma}{c^2}\,\big|D_0\psi\big|^2 + \big|\vec{D}\psi\big|^2 + \tfrac{\gamma^2}{2c^4}\,\big(\partial_0 N\big)^2 + \tfrac{\gamma}{2c^2}\,\big(\vec{\nabla}N\big)^2 + V \Big\}\, d^2x, \tag{5.4}$$

where we used the obvious notations $E_i = F_{0i}$, $B = F_{12}$ and we have assumed that the surface terms,

$$\tfrac{\gamma}{c^2}\,\vec{\nabla}\cdot\big(a_0\vec{E}\big) + \mu\vec{\nabla}\times\big(a_0\vec{a}\big), \tag{5.5}$$

fall off sufficiently rapidly at infinity. To get finite energy, we require that the energy density go to zero at infinity. Note that $|D_0\psi|^2$ does *not* go to zero at infinity, because $J_0 = (-i)\big(D_0\psi\psi^* - \psi(D_0\psi)^*\big)$ has to go to $J_0^T \neq 0$ at spatial infinity. This term combines rather with the last two terms in the potential. At spatial infinity, the energy density becomes the sum of positive terms. Requiring that all these terms go to zero allows us to conclude that finite energy requires

$$|\vec{E}| \to 0, \qquad B \to 0, \qquad |\psi|^2 \to \frac{\mathrm{I}}{2m\gamma}, \qquad N \to 0. \tag{5.6}$$

Using the Bogomolny trick and the Gauss' law as in Eqn. (5.3), the term linear in N in the potential gets absorbed. Then the energy is re-written, for the particular value $\beta = 1$, as

$$\begin{aligned} H_R = \int \Big\{ & \tfrac{\gamma}{2c^2}\,\big[\vec{E} + \vec{\nabla}N\big]^2 + \tfrac{1}{2}\big[B + \epsilon(|\psi|^2 - 2|\mu|N - \frac{\mathrm{I}}{2m\gamma})\big]^2 \\ & + \tfrac{\gamma}{c^2}\big|D_0\psi + i(N + mc^2)\psi\big|^2 + \big|(D_1 + i\epsilon D_2)\psi\big|^2 + \tfrac{\gamma^2}{2c^4}\big[\partial_0 N\big]^2 \Big\}\, d^2x \\ & - \epsilon\big(2|\mu|mc^2 - \frac{\mathrm{I}}{2m\gamma}\big)\underbrace{\int B\, d^2x}_{\text{flux}}, \end{aligned} \tag{5.7}$$

where ϵ is the sign of μ. The last term is topologic, labelled by the winding number, n, of ψ. Due to the presence of c^2, it seems to be reasonable to assume that the coefficient in front of the magnetic flux is positive. Then, chosing $n < 0$ for $\epsilon \equiv \mathrm{sign}(\mu) > 0$ and $n > 0$ for $\epsilon \equiv \mathrm{sign}(\mu) < 0$ respectively, the energy admits hence the "Bogomolny" bound

$$H_R \geq \big(2|\mu|mc^2 - \frac{\mathrm{I}}{2m\gamma}\big)\, 2\pi|n|. \tag{5.8}$$

The absolute minimum is attained by those configurations which solve the "Bogomolny" equations

$$
\begin{aligned}
&\partial_0 N = 0,\\
&\vec{\nabla} N + \vec{E} = 0,\\
&D_0\psi + i(N + mc^2)\psi = 0, \qquad (5.9)\\
&\big(D_1 + i\epsilon D_2\big)\psi = 0,\\
&B = \epsilon\big(\frac{\mathrm{I}}{2m\gamma} - |\psi|^2 + 2|\mu|N\big).
\end{aligned}
$$

It can also be checked directly that the solutions of the self-duality equations (5.9) solve the second-order field equations (5.3), when the gauge fields are static and the matter field is of the form

$$\psi = e^{-imc^2t} \times (\text{static}),$$

cf. Ref. [3]. These equations are similar to those of by Lee et al., and could be studied numerically as in Ref. [13]. Note that, just like in the case studied by Donatis and Iengo [15], the solutions are *chiral* in that the winding number and the sign of μ are correlated.

Let us stress that for getting a non-zero electrical field, the presence of a non-vanishing auxiliary field N is essential. For $N = 0$ we get rather a self-dual extension of the model of Paul and Khare [16], whose vortex solutions are purely magnetic.

Now we show that the non-relativistic limit of our relativistic model presented above is precisely the Manton model. To see this, let us set

$$\psi = \frac{1}{\sqrt{2m}} e^{-imc^2t}\,\phi. \qquad (5.10)$$

The transport current is the long-distance limit of the supercurrent, ${J^T}_\mu = \lim_{r\to\infty} J_\mu$. But $\lim_{c\to\infty} J_0/c^2 = -|\phi|^2$, so we have

$$\lim_{c\to\infty} {J^T}_0/c^2 = -\lim_{r\to\infty}|\phi|^2 = -\lim_{c\to\infty}\frac{\mathrm{I}}{\gamma} \equiv -\alpha. \qquad (5.11)$$

Then the standard procedure (described, e. g., in [3]), yields, after dropping the term $mc^2\mathrm{I}$, the non-relativistic expression

$$
\begin{aligned}
\mathcal{L}_{NR} = &-\frac{1}{2}B^2 + \gamma\frac{i}{2}\big(\phi^* D_t\phi - \phi(D_t\phi)^*\big) - \frac{1}{2m}\big|\vec{D}\phi\big|^2\\
&+\mu\big(Ba_t + E_2a_1 - E_1a_2\big) - \gamma a_t - \vec{a}\cdot\vec{J}^T \qquad (5.12)\\
&-\Big\{\frac{\beta}{8m}\big(\alpha - |\phi|^2 + 4m|\mu|N\big)^2 - \gamma\big(\alpha - |\phi|^2\big)N\Big\}.
\end{aligned}
$$

Note that there is no kinetic term left for the auxiliary field N. It can therefore be eliminated altogether by using its equation of motion,

$$4\mu^2\beta N = \big(\gamma - \frac{|\mu|\beta}{m}\big)\big(\alpha - |\phi|^2\big). \tag{5.13}$$

Inserting this into the potential, this latter becomes

$$\big(\frac{\gamma}{4|\mu|m} - \frac{\gamma^2}{8\mu^2\beta}\big)\big(\alpha - |\phi|^2\big)^2. \tag{5.14}$$

For $\alpha = 1$ and $m = 1$ in particular, we get precisely the Manton Lagrangian (1.1) with

$$\lambda = \frac{2\gamma}{|\mu|} - \frac{\gamma^2}{\mu^2\beta}. \tag{5.15}$$

The non-relativistic limit of the equations of movement (5.3) is (1.2-4), as it should be.

• In Ampère's law, the first term $(\gamma/c^2)\partial_0 F_{0i}$ can be dropped; setting (5.10), the relativistic current becomes the non-relativistic expression $\vec{J} = (1/2i)\big(\phi^*\vec{D}\phi - \phi(\vec{D}\phi)^*\big)$;

• In Gauss' law, the first term $(\gamma/c^2)\partial_i F_{0i}$ can be dropped; the time-component of the currents behave, as already noticed, as

$$\lim_{c\to\infty} J_0/c^2 = -|\phi|^2, \qquad \text{and} \qquad \lim_{c\to\infty} {J^T}_0/c^2 = -\alpha = -1.$$

• In the equation for the auxiliary field N the first term $(\gamma/c^2)\partial_\mu\partial^\mu N$ can be dropped and the $c \to \infty$ limit of $\partial V/\partial N = 0$ is (5.13);

• Finally, setting (5.10) in the nonlinear Klein-Gordon equation and using the equation of motions (5.13) for N, a lengthy but straightforward calculation yields the non-linear Schrödinger equation (1.2), as expected.

Note also that, for the self-dual value $\beta = 1$ (when λ in (5.15) becomes (2.3)), the non-relativistic limit of the (relativistic) self-dual equations (5.9) fixes a_0 and N as

$$a_0 = N = \big(-\frac{\epsilon}{4\mu} + \frac{\gamma}{4\mu^2}\big)\big(1 - |\phi|^2\big). \tag{5.16}$$

which is consistent with Eq. (2.4). The other equations reduce in turn to our non-relativistic self-dual equations (2.2).

6. Further models

A. As already said, some of our formulæ bear a strong ressemblence to those of Barashenkov and Harin [7], who study the system described by

$$\begin{aligned}\mathcal{L} = \frac{1}{2}E^2 - \frac{1}{2}B^2 + \frac{i}{2}\big(\phi^* D_t\phi - \phi(D_t\phi)^*\big) - \frac{1}{2}\big|\vec{D}\phi\big|^2 - \frac{\lambda}{8}\big(1 - |\phi|^2\big)^2 \\ + \mu\big(Ba_t + E_2 a_1 - E_1 a_2\big) - \gamma a_t.\end{aligned} \tag{6.1}$$

This Lagrangian only differs from the Manton model in that it contains the full Maxwell term, while the transport term $\vec{J}^T \cdot \vec{a}$ is missing. The Barashenkov-Harin model has, therefore, no clear symmetry: the (full) Maxwell term is Lorentz invariant; the matter term is Galilei invariant; the Chern-Simons term is invariant with respect to any diffeomorphism. Finally, their naked $-\gamma \cdot a_t$ term breaks both the Lorentz and Galilei invariance.

The energy of the Barashenkov-Harin model is

$$H = \int \left\{ \tfrac{1}{2}\vec{E}^2 + \tfrac{1}{2}B^2 + \tfrac{1}{2}\big|\vec{D}\phi\big|^2 + \frac{\lambda}{8}\big(1 - |\phi|^2\big)^2 \right\} d^2\vec{x}, \tag{6.2}$$

while their Gauss law reads

$$\vec{\nabla} \cdot \vec{E} - 2\mu B - |\phi|^2 + \gamma = 0. \tag{6.3}$$

Now the presence of $\vec{E}^2$ in the energy and of $\vec{\nabla} \cdot \vec{E}$ in their Gauss' law only allows, just like in other relativistic models, a vanishing electric field, [17] — unless an auxiliary field is added [13], [18].

After putting the electric field to zero by hand, the remaining Barashenkov-Harin equations coincide with ours. It is hence precisely the *absence* of the electric terms — dictated by the requirement of a *consistently non-relativistic theory* — which opens the door for solutions with nonzero electric field in Manton's model.

Let us note in conclusion that the more general type of self-duality with two non-vanishing components [19] only works in the pure Chern-Simons case, and breaks down when the Maxwell term is present, due to $\vec{\nabla} \times B$ in Ampère's law.

<u>B</u>. Let us mention that a consistently non-relativistic Maxwell-Chern-Simons model has also been considered before, namely in a "non-relativistic Kaluza-Klein-type" framework [20]. There one starts with a four-dimensional (relativistic) coupled Maxwell-Chern-Simons theory. When the theory is reduced to 2+1 dimensions by factoring out a lightlike, covariantly-constant direction, one gets a non-relativistic system with equations of motion

$$\begin{aligned} &iD_t\phi = -\frac{1}{2}\vec{D}^2\phi + \frac{\delta U}{\delta\phi^*}, \\ &\epsilon_{ij}\partial_j B = J_i + 2\mu\,\epsilon_{ij}\,E_j, \\ &2\mu B = -|\phi|^2, \end{aligned} \tag{6.4}$$

Note the absence of the transport current in Ampère's law and that the Gauss law has the Jackiw-Pi form. This system can be solved along the same lines as in Manton's case: Using the Gauss law, the self-duality equations

$$\begin{aligned} &(D_1 \pm iD_2)\phi = 0, \\ &2\mu B = -|\phi|^2, \end{aligned} \tag{6.5}$$

are readily seen to solve the field equations, provided the potential is

$$U(\phi) = -\frac{\lambda}{8}\,|\phi|^4, \qquad \lambda \equiv \tfrac{1}{\mu^2} \mp \tfrac{2}{\mu}. \tag{6.6}$$

Then, inserting

$$\vec{a} = \pm\tfrac{1}{2}\vec{\nabla}\times\log\varrho + \vec{\nabla}\omega \qquad \text{and} \qquad a_t = \tfrac{1}{4}\big(\pm\tfrac{1}{\mu} - \tfrac{1}{\mu^2}\big)\varrho, \tag{6.7}$$

into the Gauss' law, we get the Liouville equation,

$$\triangle\log\varrho = \pm\frac{1}{\mu}\,\varrho.$$

Regular solutions arise when the r. h. s. is negative. Hence the upper sign works for $\mu < 0$ and the lower sign works for $\mu > 0$. For both signs, the particle density $\varrho = |\psi|^2$ satisfies finally

$$\triangle\log\varrho = -\frac{1}{|\mu|}\,\varrho, \tag{6.8}$$

which is precisely the problem solved by Jackiw and Pi in the pure Chern-Simons case [3]. Note that $\lambda = 1/\mu^2 \pm 2/\mu$ is always positive so that the potential (6.6) is attractive.

The same conclusion can be reached by noting that the energy of this system is simply

$$H = \int\Big\{\frac{1}{2}\big|\vec{D}\phi\big|^2 - \frac{g}{2}|\phi|^4\Big\}\,d^2\vec{x}, \qquad g = \frac{\lambda}{4} - \frac{1}{4\mu^2}, \tag{6.9}$$

which is again of the Jackiw-Pi form, the only effect of the Maxwell field being a shift, $\lambda \to \lambda - 1/\mu^2$, in the coefficient of the non-linearity. The latter model is known however to be self-dual precisely when the coefficient of the Chern-Simons term and the non-linearity are related as $g = \mp 1/2\mu > 0$, which yields the value (6.6) for λ once again.

Note that this system admits the full "geometric" Schrödinger symmetry (3.7), just like in the Jackiw-Pi case. The "conformal" symmetry is indicated by the energy-momentum tensor satisfying now $\sum_i T^{ii} = 2T^{00}$, and then the same argument as in the Jackiw-Pi case shows that all static solutions are necessarily self-dual [3].

Let us point out in conclusion that the Manton model is in fact the non-relativistic field theoretical generalization of the static system introduced by Girvin [21] in his "Landau-Ginzburg" theory for the Quantum Hall Effect.

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Appendix : Dirac brackets

Using the expression (3.3) of the momenta and setting $\phi = f e^{i\omega}$, the Dirac bracket $\{\mathcal{P}_1, \mathcal{P}_2\}$ becomes

$$\{\mathcal{P}_1, \mathcal{P}_2\} = \gamma \int \vec{\nabla} \times \big[f^2(\vec{\nabla}\omega - \vec{J}^T)\big] d^2\vec{x} = \gamma \oint_S \big(\vec{\nabla}\omega - \vec{J}^T\big) \cdot d\vec{\ell} \tag{A.1}$$

by using Stokes' theorem, where S is the circle at infinity. Now if $\vec{J}^T = 0$ then the current $\vec{J}$ goes to zero at infinity. $\vec{A}$ is hence a pure gauge and the integral in A.1 yields (2π times) the winding number,

$$\{\mathcal{P}_1, \mathcal{P}_2\} = 2\pi n\,\gamma. \tag{A.2}$$

For $\vec{J}^T \neq 0$, a boost with velocity $\vec{\beta} = \vec{J}^T/\gamma$ absorbes the transport current into the phase and we are back in the previous case. The result (A.2) is hence valid in all cases.

The conserved quantities, denoted by $\vec{p}$ and $\vec{g}$, associated to "hidden boosts and translations" are readily found by inserting (3.10) into (3.12). Then a lengthy but elementary calculation yields the commutation relations (3.18). For example,

$$\begin{aligned}\{g_1, p_1\} = \gamma \int \Big(&\cos^2\omega t\, x_1\partial_1(|\Phi|^2 - 1) + \sin\omega t\cos\omega t\, x_2\partial_1(|\Phi|^2 - 1) \\ &+ \sin\omega t\cos\omega t\, x_1\partial_2(|\Phi|^2 - 1) + \sin^2\omega t\, x_2\partial_2(|\Phi|^2 - 1)\Big) d^2\vec{x}.\end{aligned} \tag{A.3}$$

Integrating by parts yields now, *using that* $|\Phi| \to 1$ *at infinity*, yields the last relation in (3.18), with the particle number N being defined by Eq. (2.10). It is worth mentionning that these commutation relations are in fact the remnants of those of the centrally extended Galilei group.

References

[1] N. Manton, Ann. Phys. (N. Y.). **256**, 114 (1997).

[2] M. Le Bellac and J.-M. Lévy-Leblond, *Il Nuovo Cimento* **14B**, 217 (1973).

[3] R. Jackiw and S-Y. Pi, *Phys. Rev. Lett.* **64**, 2969 (1990); *Phys. Rev.* **D42**, 3500 (1990). For reviews, see R. Jackiw and S-Y. Pi, *Prog. Theor. Phys. Suppl.* **107**, 1 (1992), or G. Dunne, *Self-Dual Chern-Simons Theories.* Springer Lecture Notes in Physics. New Series: Monograph 36. (1995).

[4] A. A. Abrikosov, *Sov. Phys. JETP* **5**, 1174 (1957). The Abrikosov vortices have been imbedded into the relativistic model of H. B. Nielsen and P. Olesen, *Nucl. Phys.* **B61**, 45 (1973).

[5] E. B. Bogomolny, *Sov. J. Nucl. Phys.* **24**, 449 (1976); H. J. De Vega and F. A. Schaposnik, *Phys. Rev.* **D14**, 1100 (1976).

[6] E. Weinberg, *Phys. Rev.* **D19**, 3008 (1979); C. H. Taubes, *Commun. Math. Phys.* **72**, 277 (1980).

[7] I. Barashenkov and A. Harin *Phys. Rev. Lett.* **72**, 1575 (1994); *Phys. Rev.* **D52**, 2471 (1995).

[8] P. Forgács and N. Manton, *Commun. Math. Phys.* **72**, 15 (1980). A similar definition has appeared in R. Jackiw, *Phys. Rev. Lett.* **41**, 1635 (1979). The associated conservation laws are studied in R. Jackiw and N. Manton, *Ann. Phys.* (N. Y.) **127**, 257 (1980). See also R. Jackiw, *Acta Physica Austr.* (Suppl.) **22**, 383 (1980).

[9] Z. F. Ezawa, M. Hotta and A. Iwazaki, *Phys. Rev. Lett.* **67**, 411 (1991), *Phys. Rev.* **D44**, 452 (1991). See also R. Jackiw and S-Y. Pi, *Phys. Rev. Lett.* **67**, 415 (1991); *Phys. Rev.* **D44**, 2524 (1991); M. Hotta *Prog. Theor. Phys.* **86**, 1289 (1991).

[10] C. Duval, P. A. Horváthy and L. Palla, *Phys. Rev.* **D50**, 6658 (1995).

[11] U. Niederer, *Helvetica Physica Acta* **46**, 191 (1973).

[12] C. Duval, P. A. Horváthy and L. Palla, *Phys. Rev.* **D52**, 4700 (1995); Ann. Phys. (N. Y.) **249**, 265 (1996).

[13] C. Lee, K. Lee, H. Min, *Phys. Lett.* **B252**, 79 (1990).

[14] K. Lee, *Phys. Rev.* **D49**, 4265 (1994); K. Lee and P. Yi, *Phys. Rev.* **D52**, 2412 (1995).

[15] P. Donatis and R. Iengo, *Phys. Lett.* **B320**, 64 (1994).

[16] S. K. Paul and A. Khare, *Phys. Lett.* **B174**, 420 (1986).

[17] B. Julia, A. Zee, *Phys. Rev. Lett.* **11**, 2227 (1975).

[18] G. Dunne and C. Trugenberger, *Phys. Rev.* **D43**, 1323 (1991).

[19] L. Martina, O. K. Pashaev, G. Soliani, *Phys. Rev.* **B48**, 15 787 (1993), Z. Németh, Budapest preprint (1997).

[20] C. Duval, P. A. Horváthy and L. Palla, *Phys. Lett.* **B325**, 39 (1994).

[21] S. M. Girvin in *The Quantum Hall Effect*, edited by R. E. Prange and S. M. Girvin (Springer Verlag, N. Y. 1986), Chapt. 10. See also S. M. Girvin and A.-H. Mac Donald, *Phys. Rev. Lett.* **58**, 303 (1987).